ONE STANDARD DEVIATION DOES NOT SUFFICE: A LOWER BOUND FOR DISCREPANCY

Abstract. In this short note, we prove that for each sufficiently large $n \in \mathbb{N}$ admitting an $n \times n$ Hadamard matrix, there exists $A = A_n \in \{\pm 1\}^{n\times n}$ such that

$$\operatorname{disc}(A) \geqslant (1+\delta)\sqrt{n}, \quad \delta = 10^{-20}, \quad \operatorname{disc}(A) := \inf_{x \in \{\pm 1\}^n} \|Ax\|_\infty .$$

In particular, this implies that

$$\limsup_{n\to\infty} \sup_{A \in \{\pm 1\}^{n\times n}} \frac{1}{\sqrt{n}} \operatorname{disc}(A) \geqslant 1+\delta > 1 ,$$

thereby proving [BDL+26, Conjecture 12]. The crux of our construction is a probabilistic argument based on a random modification of Hadamard matrix. The main result of this paper were generated using GPT-6 Astra.

Contents

1. Introduction and main results

Given $n \in \mathbb{N}$ and $A \in \mathbb{R}^{n\times n}$, we define the discrepancy of A as $\operatorname{disc}(A) := \inf_{x\in\{\pm 1\}^n} \|Ax\|_\infty$. One of the motivations of this question is to understand how much smaller is $\operatorname{disc}(A)$ compared to the value, e.g. in expectation, when x is drawn uniformly from the hypercube. In 1985, Joel Spencer [Spe85] showed the celebrated "Six Deviations Suffice" Theorem, which states that for any positive integer n and any $A \in \{\pm 1\}^{n\times n}$ we have $\operatorname{disc}(A) \leqslant 6\sqrt{n}$. There have since been improvements on this constant, but the question we will pursue here concerns lower bounds. In fact, as discussed in [BDL+26], there seems to be no known construction (or proof of existence) of an infinite family achieving a lower bound strictly larger than one. This motivated them formulate the following conjecture (see Conjecture 12 therein).

Conjecture 1.1. *We have*

$$\limsup_{n\to\infty} \sup_{A \in \{\pm 1\}^{n\times n}} \frac{1}{\sqrt{n}} \operatorname{disc}(A) \geqslant 1+\delta > 1 ,$$

The main goal of this short note is to prove Conjecture 1.1. Recall that a matrix $H_n \in \{\pm 1\}^{n\times n}$ is a Hadamard matrix if $H_n^\top H_n = nI_n$.

Theorem 1.2. *There exists an absolute constant N_0 such that the following holds. For each even $n \geqslant N_0$ admitting an $n \times n$ Hadamard matrix H_n, there exists $A_n \in \{\pm 1\}^{n\times n}$ such that $\operatorname{disc}(A_n) \geqslant (1+\delta)\sqrt{n}$, where $\delta = 10^{-20} > 0$.*

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In particular, since it is well known that an $n \times n$ Hadamard matrix exists for all $n = 2^k, k \geqslant 1$, Theorem 1.2 immediately implies Conjecture 1.1.

Our proof is based on a probabilistic argument that randomly modifies the signs of the Hadamard matrix H_n. More precisely, given an $n \times n$ Hadamard matrix H, independently for each row i, we choose a uniform subset $S_i \subseteq [n]$ of cardinality $r = \lfloor \delta n \rfloor$. Set

$$A_{i,j} = \begin{cases} H_{i,j}, & j \notin S_i\,; \\ -H_{i,j}, & j \in S_i\,. \end{cases} \tag{1.1}$$

It suffices to show the following result.

Theorem 1.3. *For all sufficiently large n and every $n \times n$ Hadamard matrix H, the random matrix A defined in (1.1) with $\delta = 10^{-20}$ satisfies*

$$\mathbb{P}\big(\mathrm{disc}(A) \leqslant (1+\delta)\sqrt{n}\big) \leqslant e^{-n/1600} + e^{-(1-\log 2)n}\,.$$

It is clear that Theorem 1.3 implies Theorem 1.2.

Statement on AI use. The main results in this paper were generated by GPT-6 Astra, prompted by Zhangsong Li without any mathematical input.

2. Proof of the main theorem

In this section, we prove Theorem 1.3. We first record a number of probabilistic and combinatorial inputs that will be used in the proof. The proof of those inputs are postponed to Section 3. We first record a bound for the number of "nearly flat" images of Hadamard matrices.

Lemma 2.1. *Let $H \in \{-1,1\}^{n \times n}$ satisfy $H^\top H = nI$, where $n \geqslant 2$. Then*

$$\#\mathcal{F}(H) := \#\left\{x \in \{-1,1\}^n : \left\|\frac{1}{\sqrt{n}}Hx\right\|_1 > \frac{9}{10}n\right\} \leqslant 2^n e^{-n/800}\,.$$

We then record two facts for a random r-subset $S \subseteq [n]$.

Lemma 2.2. *Let $1 \leqslant r \leqslant n$ be an integer, let $w_1, \dots, w_n \in \{-1,1\}$, let S be a uniform r-subset of $[n]$, and let $B = \sum_{j \in S} w_j$. Then, for $u \geqslant 0$,*

$$\mathbb{P}(B - \mathbb{E}B \geqslant u) \leqslant e^{-u^2/(2r)}.$$

The same estimate holds for the lower tail.

Lemma 2.3. *Fix $\delta \in (0,1)$ and a compact interval $\Gamma \subset \mathbb{R}$. Let $r = \lfloor \delta n \rfloor$, and suppose that $w \in \{-1,1\}^n$ satisfies $\sum_j w_j = v\sqrt{n}$ with $v \in \Gamma$. Let S be a uniform r-subset of $[n]$, and put $Y = n^{-1/2}(\sum_j w_j - 2\sum_{j \in S} w_j)$. There exists $\varepsilon_n = \varepsilon_n(\Gamma, \delta) \geqslant 0$ tending to zero such that, uniformly over all such w,*

$$\sup_{T \in \mathbb{R}} \left| \mathbb{P}(Y \leqslant T) - \Phi\left(\frac{T - (1-2\delta)v}{2\sqrt{\delta(1-\delta)}}\right) \right| \leqslant \varepsilon_n.$$

Here Φ is the standard normal distribution function.

For $v \in \mathbb{R}$ such that $v\sqrt{n} \in \{-n, -n+2, \dots, n\}$, choose $w \in \{-1,1\}^n$ with $\sum_j w_j = v\sqrt{n}$ and define

$$p_n(v) = \mathbb{P}\left(\left|\sum_j w_j - 2\sum_{j \in S} w_j\right| \leqslant (1+\delta)\sqrt{n}\right), \qquad |S| = \lfloor \delta n \rfloor,$$

where S is uniformly chosen. This probability depends only on v, not on the particular choice of w.

Lemma 2.4. *Set $\delta = 10^{-20}$, $\eta = 1/1600$, and $\lambda = 1/\sqrt{8\pi\delta}$. For all sufficiently large n and every $v \in \mathbb{R}$ satisfying $v\sqrt{n} \in \{-n, -n+2, \dots, n\}$, we have*

$$p_n(v) \leqslant \exp\left(-\log 2 + \eta - \lambda(v^2 - 1)\right). \tag{2.1}$$

We now provide the proof of Theorem 1.3.

Proof of Theorem 1.3. Put $t = 1 + \delta$, $\delta_n = r/n$, and $a_n = 1 - 2\delta_n$, and take n to be sufficiently large that $r \geqslant 1$ and Lemma 2.4 applies. Thus $\delta_n \leqslant \delta$ and $a_n \geqslant 1 - 2\delta$. We use $\eta = 1/1600$ and $\lambda = 1/\sqrt{8\pi\delta}$ as in Lemma 2.4. We will apply a union bound to control $\|Ax\|_\infty$ for all $x \in \{\pm 1\}^n$. Fix a sign vector x and write $v = Hx/\sqrt{n}$. Orthogonality gives

$$\sum_i v_i^2 = n. \tag{2.2}$$

We divide into two cases:

Case 1: $x \in \mathcal{F}(H)$. The random choices in different rows of A are independent, so Lemma 2.4 and (2.2) imply the uniform bound

$$\begin{aligned}\mathbb{P}(\|Ax\|_\infty \leqslant t\sqrt{n}) &= \prod_i p_n(|v_i|) \\ &\leqslant \exp\left((-\log 2 + \eta)n - \lambda \sum_i (v_i^2 - 1)\right) = 2^{-n} e^{\eta n}.\end{aligned}$$

Combined with Lemma 2.1 and a union bound, this gives

$$\begin{aligned}&\mathbb{P}\left(\|Ax\|_\infty \leqslant t\sqrt{n} \text{ for some } x \in \mathcal{F}(H)\right) \\ &\qquad \leqslant 2^n e^{-n/800} \cdot 2^{-n} e^{\eta n} = e^{-n/1600}.\end{aligned} \tag{2.3}$$

Case 2: $x \notin \mathcal{F}(H)$. Then $\sum_i |v_i| \leqslant 0.9n$. Lemma 2.2 gives

$$\begin{aligned}\mathbb{P}(\|Ax\|_\infty \leqslant t\sqrt{n}) &= \prod_i p_n(|v_i|) \\ &\leqslant \prod_{i=1}^n \exp\left(-\frac{(a_n|v_i| - t)_+^2}{8\delta_n}\right).\end{aligned} \tag{2.4}$$

Set $u_i = (a_n|v_i| - t)_+$. For $w \geqslant 0$ one has $w^2 \leqslant tw + w(w-t)_+$. Consequently, by Cauchy–Schwarz and (2.2),

$$\begin{aligned}a_n^2 n &\leqslant t a_n \sum_i |v_i| + a_n \sum_i |v_i| u_i \\ &\leqslant \frac{9}{10} t a_n n + a_n\sqrt{n}\left(\sum_i u_i^2\right)^{1/2}.\end{aligned}$$

Since $a_n - 9t/10 \geqslant 1/10 - 29\delta/10 > 0$, division by $a_n\sqrt{n}$, rearrangement, and squaring yield

$$\sum_i u_i^2 \geqslant \left(a_n - \frac{9}{10}t\right)^2 n \geqslant \left(\frac{1}{10} - \frac{29}{10}\delta\right)^2 n.$$

Substituting this estimate into (2.4) and applying a union bound, we obtain

$$\begin{aligned}&\mathbb{P}\left(\|Ax\|_\infty \leqslant t\sqrt{n} \text{ for some } x \notin \mathcal{F}(H)\right) \\ &\qquad \leqslant 2^n \exp\left(-\frac{(1/10 - (29/10)\delta)^2 n}{8\delta}\right) \leqslant e^{-(1-\log 2)n}.\end{aligned} \tag{2.5}$$

Combining (2.3) and (2.5) proves the result. □

3. Preliminary inputs

In this section, we complete the omitted proofs in Lemmas 2.1–2.4.

3.1. Proof of Lemma 2.1.

Proof. Let $Q = H/\sqrt{n}$, let X be uniform on the sign cube, and put $F(x) = \|Qx\|_1$. Orthogonality of any two columns of H implies that n is even. Write $n = 2m$. An elementary binomial summation, together with $\binom{2m}{m}/4^m \leqslant 1/\sqrt{\pi m}$, gives

$$\mathbb{E}F(X) = \sqrt{n}\,\mathbb{E}\left|\sum_{j=1}^{n} X_j\right| = \sqrt{n}\,2m\,\frac{\binom{2m}{m}}{4^m} \leqslant \sqrt{\frac{2}{\pi}}\,n < \frac{4}{5}n.$$

We next prove the concentration estimate needed here. Denote by $x^{(j)}$ the vector obtained by reversing the jth coordinate of x. Choose $y_i \in \{-1, 1\}$ such that $y_i(Qx)_i = |(Qx)_i|$; either sign may be used when $(Qx)_i = 0$. The vector $g = Q^\top y$ is a subgradient of F at x, and $\|g\|_2^2 = n$. Convexity therefore gives

$$F(x) - F(x^{(j)}) \leqslant 2x_j g_j, \qquad \sum_{j=1}^{n}(F(x) - F(x^{(j)}))_+^2 \leqslant 4n\,.$$

Talagrand's concentration inequality then implies that

$$\mathbb{P}(F(X) \geqslant \mathbb{E}F(X) + u) \leqslant e^{-u^2/(8n)}.$$

Taking $u = n/10$ proves the claimed counting bound. □

3.2. Proof of Lemma 2.2.

Proof. If $r = n$, then B is deterministic and the conclusion is immediate. Suppose that $r < n$. Generate S by sampling distinct indices $J_1, \dots, J_r$ uniformly without replacement. Let $\mathcal{G}_k = \sigma(J_1, \dots, J_k)$ and $M_k = \mathbb{E}[B \mid \mathcal{G}_k]$, so that $M_0 = \mathbb{E}B$ and $M_r = B$. For $1 \leqslant k \leqslant r$, let $m_{k-1} = (n-k+1)^{-1}\sum_{j\notin\{J_1,\dots,J_{k-1}\}} w_j$ be the mean of the remaining signs. A direct computation gives $M_k - M_{k-1} = (n-r)(w_{J_k} - m_{k-1})/(n-k)$. Conditionally on $\mathcal{G}_{k-1}$, this increment has mean zero and is supported on an interval of length at most $2(n-r)/(n-k) \leqslant 2$.

For any centered random variable Z supported on an interval of length at most two, one has $\mathbb{E}e^{\theta Z} \leqslant e^{\theta^2/2}$ for every $\theta \in \mathbb{R}$. Indeed, the second derivative of $\log \mathbb{E}e^{\theta Z}$ is the variance under the corresponding exponentially tilted law, which is at most one because its support remains in the same interval. Integrating twice and using $\mathbb{E}Z = 0$ proves the bound. Applying this conditionally to the martingale increments and iterating gives $\mathbb{E}e^{\theta(B-\mathbb{E}B)} \leqslant e^{r\theta^2/2}$. For $u > 0$, Markov's inequality with $\theta = u/r$ yields

$$\mathbb{P}(B - \mathbb{E}B \geqslant u) \leqslant \exp\left(-\theta u + \frac{r\theta^2}{2}\right) = \exp\left(-\frac{u^2}{2r}\right).$$

The case $u = 0$ is immediate. Replacing every w_j by $-w_j$ gives the lower-tail estimate. □

3.3. **Proof of Lemma 2.3.**

Proof. Let $K = \#\{j : w_j = 1\} = (n + v\sqrt{n})/2$, put $p = K/n$ and $\delta_n = r/n$, and let $X = \#\{j \in S : w_j = 1\}$. Then $p = 1/2 + v/(2\sqrt{n})$ and

$$\mathbb{P}(X = k) = \frac{\binom{K}{k}\binom{n-K}{r-k}}{\binom{n}{r}}, \qquad Y = (1 - 2\delta_n)v - \frac{4}{\sqrt{n}}(X - rp).$$

Throughout the proof, all error estimates are uniform over the allowed $v \in \Gamma$. Since Γ is compact, $p = 1/2 + O_\Gamma(n^{-1/2})$, while $\delta_n \to \delta$. In particular, p and δ_n stay bounded away from zero and one for all sufficiently large n.

Put $\tau_n^2 = n\delta_n(1 - \delta_n)p(1 - p)$. We first establish a uniform local normal approximation. Fix $M > 0$ and consider integers k with $|k - rp| \leqslant M\sqrt{n}$. Writing $u = k - rp$, Stirling's formula gives

$$\begin{aligned}\log \mathbb{P}(X = k) = Kh\left(\delta_n + \frac{u}{K}\right) + (n - K)h\left(\delta_n - \frac{u}{n - K}\right) - nh(\delta_n)\\ - \frac{1}{2}\log(2\pi\tau_n^2) + o(1),\end{aligned}$$

where $h(s) = -s\log s - (1-s)\log(1-s)$. The error is uniform for the indicated k and v. Taylor expansion about δ_n cancels the linear terms, and $h''(s) = -1/[s(1 - s)]$ gives

$$\begin{aligned}Kh\left(\delta_n + \frac{u}{K}\right) + (n - K)h\left(\delta_n - \frac{u}{n - K}\right) - nh(\delta_n)\\ = -\frac{u^2}{2\tau_n^2} + O_{\Gamma,\delta,M}(n^{-1/2}).\end{aligned}$$

Consequently,

$$\mathbb{P}(X = k) = \frac{1 + o(1)}{\sqrt{2\pi}\,\tau_n}\exp\left(-\frac{(k - rp)^2}{2\tau_n^2}\right), \qquad |k - rp| \leqslant M\sqrt{n},$$

uniformly over k and v.

The sampling indicators give $\mathbb{E}X = rp$ and $\mathrm{Var}(X) = rp(1 - p)(n - r)/(n - 1) = \tau_n^2 n/(n - 1)$. Thus Chebyshev's inequality bounds $\mathbb{P}(|X - rp| > M\sqrt{n})$ by $C_{\Gamma,\delta}/M^2$, uniformly in v. Summing the local approximation over $|k - rp| \leqslant M\sqrt{n}$ and comparing with the corresponding Gaussian Riemann sums, uniformly in the upper endpoint, therefore yields

$$\sup_{z\in\mathbb{R}}\left|\mathbb{P}\left(\frac{X - rp}{\tau_n} \leqslant z\right) - \Phi(z)\right| \longrightarrow 0,$$

uniformly over all allowed v. Here one first lets $n \to \infty$ for fixed M and then lets $M \to \infty$; the Gaussian tails also vanish uniformly because $\tau_n/\sqrt{n}$ stays bounded above and away from zero.

Finally, uniformly over $v \in \Gamma$,

$$(1 - 2\delta_n)v \longrightarrow (1 - 2\delta)v, \qquad \frac{4\tau_n}{\sqrt{n}} = 4\sqrt{\delta_n(1 - \delta_n)p(1 - p)} \longrightarrow 2\sqrt{\delta(1 - \delta)}.$$

Using the representation of Y above, reflection symmetry of the standard normal distribution, and continuity of its distribution function proves the asserted uniform estimate. More explicitly, the uniform distribution-function approximation also holds for left limits, so reflecting $(X - rp)/\tau_n$ causes no endpoint problem. The convergence of the means and of the positive standard deviations then gives uniform convergence of the normal distribution functions for all $T \in \mathbb{R}$. □

3.4. **Proof of Lemma 2.4.**

Proof. Replacing w by $-w$ shows that $p_n(-v) = p_n(v)$, so it is enough to consider $v \geqslant 0$. Put $t = 1 + \delta$, $a = 1 - 2\delta$, and $\sigma = 2\sqrt{\delta(1-\delta)}$. Also write $\delta_n = \lfloor \delta n \rfloor / n$ and $a_n = 1 - 2\delta_n$, and take n sufficiently large that $\lfloor \delta n \rfloor \geqslant 1$. Put $B = \sum_{j \in S} w_j$ and $Y = v - 2B/\sqrt{n}$. Then $\mathbb{E}B = \delta_n v \sqrt{n}$ and $Y = a_n v - 2(B - \mathbb{E}B)/\sqrt{n}$. Since $\{|Y| \leqslant t\} \subseteq \{Y \leqslant t\}$, Lemma 2.2 gives

$$p_n(v) \leqslant \exp\left(-\frac{(a_n v - t)_+^2}{8\delta_n}\right) \leqslant \exp\left(-\frac{(av - t)_+^2}{8\delta}\right),$$

where the second inequality uses $a_n \geqslant a$ and $\delta_n \leqslant \delta$.

We also use the elementary Gaussian inequality $\log \Phi(z) \leqslant -\log 2 + \sqrt{2/\pi}\, z$ for every $z \in \mathbb{R}$. To see this, let $\varphi(z) = (2\pi)^{-1/2} e^{-z^2/2}$ and $R(z) = \varphi(z)/\Phi(z)$. Differentiation gives $R'(z) = -R(z)(z + R(z)) < 0$, since $z\Phi(z) + \varphi(z) = \int_{-\infty}^{z} (z-u)\varphi(u)\, du > 0$. Thus $\log \Phi$ is concave, and its tangent at zero is the claimed upper bound.

Case 1: $0 \leqslant v \leqslant 1 - 10\sqrt{\delta}$. In this range, $\lambda(1 - v^2) \geqslant (20 - 100\sqrt{\delta})/\sqrt{8\pi} > \log 2$. The right-hand side of (2.1) is therefore greater than one, and the conclusion follows from $p_n(v) \leqslant 1$.

Case 2: $|v - 1| \leqslant 10\sqrt{\delta}$. Apply Lemma 2.3 with the fixed compact interval $\Gamma = [1 - 10\sqrt{\delta}, 1 + 10\sqrt{\delta}]$. Uniformly over the allowed v in this interval,

$$p_n(v) \leqslant \mathbb{P}(Y \leqslant t) \leqslant \Phi\left(\frac{t - av}{\sigma}\right) + \varepsilon_n, \qquad \varepsilon_n \longrightarrow 0.$$

The Gaussian inequality and the choice $\lambda = 1/\sqrt{8\pi\delta}$ give

$$\begin{aligned}\log \Phi\left(\frac{t - av}{\sigma}\right) + \log 2 + \lambda(v^2 - 1) &\leqslant \sqrt{\frac{2}{\pi}} \frac{t - av}{\sigma} + \lambda(v^2 - 1) \\ &= \lambda(v-1)^2 + \frac{t - av - \sqrt{1-\delta}(1 - v)}{\sqrt{2\pi\delta(1-\delta)}}.\end{aligned}$$

The numerator in the last fraction equals $1 + \delta - \sqrt{1-\delta} + v(\sqrt{1-\delta} - 1 + 2\delta)$, which lies between zero and $2\delta(1 + v) \leqslant 6\delta$. Moreover, $\lambda(v-1)^2 \leqslant 25\sqrt{\delta}$ and $\sqrt{2\pi\delta(1-\delta)} \geqslant 2\sqrt{\delta}$. The last display is consequently at most $30\sqrt{\delta} < \eta/2$. Hence

$$p_n(v) \leqslant \exp\left(-\log 2 + \frac{\eta}{2} - \lambda(v^2 - 1)\right) + \varepsilon_n.$$

Since $e^{-\log 2 - \lambda(v^2 - 1)} \geqslant e^{-6}$ throughout this interval, taking n so large that $\varepsilon_n \leqslant e^{-6}(e^{\eta} - e^{\eta/2})$ proves (2.1) uniformly in this case.

Case 3: $v \geqslant 1 + 10\sqrt{\delta}$. Put $u = v - 1 \geqslant 10\sqrt{\delta}$. Since $\delta = 10^{-20}$, one has $av - t = (1 - 2\delta)u - 3\delta \geqslant 3u/4$ and $\lambda \leqslant 1/(4\sqrt{\delta})$. Therefore

$$\begin{aligned}\frac{(av - t)^2}{8\delta} - \lambda(v^2 - 1) &\geqslant \frac{9u^2}{128\delta} - \frac{u^2 + 2u}{4\sqrt{\delta}} \\ &\geqslant \frac{u^2}{16\delta} - \frac{u}{2\sqrt{\delta}} \geqslant \frac{5}{4} > \log 2.\end{aligned}$$

The second inequality uses $\sqrt{\delta} \leqslant 1/32$, and the third uses $u/\sqrt{\delta} \geqslant 10$. The sampling tail bound at the start of the proof now gives $p_n(v) \leqslant \exp(-\log 2 - \lambda(v^2 - 1))$, which is stronger than (2.1). The three cases exhaust all nonnegative v, and symmetry completes the proof. $\square$

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