RANDOM 3-REGULAR GRAPHS ARE NOT GLOBALLY SYNCHRONIZING

ABSTRACT. We prove that for a uniform random 3-regular graph G on n vertices, the probability that G is globally synchronizing tends to 0 as $n \to \infty$. This refutes [BDL+26, Conjecture 3] in a strong form. The main result of this paper was obtained using GPT6-Astra.

CONTENTS

1. INTRODUCTION AND MAIN RESULTS

In this short note, we will focus on spontaneous synchronization of coupled oscillators with pairwise connections given by a graph with adjacency matrix $A \in \mathbb{R}^{n\times n}$. The celebrated Kuramoto model [Kur05] models the oscillators as gradient flow on

$$\mathcal{E}(\theta) = \frac{1}{2}\sum_{i,j=1}^{n} A_{i,j}(1-\cos(\theta_i-\theta_j)) \tag{1.1}$$

where the parameterization $\theta \in [0, 2\pi)^n$ represents each oscillators angle in the moving frame of its natural frequency. For the detailed derivation of this potential, see, e.g. [ABK+26]. This motivates the following Definition which is central to what follows. We say A is globally synchronizing if the only local minima of $\mathcal{E}(\theta)$ are the global minima corresponding to $\theta_i = c, \forall i$. A graph G is said to be globally synchronizing if its adjacency matrix is globally synchronizing.

[ABK+26] recently showed that a random Erdős-Rényi graph is globally synchronizing above the connectivity threshold with high probability; The same paper also shows that a uniform d-regular graph is globally synchronizing with high probability for $d \geqslant 600$, but it leaves open the following question.

Conjecture 1.1. *For even n, denote by $\mathcal{G}_3(n)$ the uniform measure over all 3-regular graphs on n vertices. Then for $G \sim \mathcal{G}_3(n)$, we have*

$$\lim_{n\to\infty} \mathbb{P}(G \text{ is globally synchronizing}) = 1\,.$$

The goal of this short note is to refute Conjecture 1.1 in a strong form.

Theorem 1.2. *Let G be uniform among the simple 3-regular graphs on $[n]$, with n even. Then*

$$\lim_{n\to\infty} \mathbb{P}(G \text{ is globally synchronizing}) = 0\,.$$

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Our obstruction actually only relies two ingredient from the random regular graph: a spectral gap estimate and the Poisson limit of small cycles. Precisely, let $\mathcal{L}_G = 3I_n - A$ be the ordinary, unnormalized Laplacian of a 3-regular graph $G = (V, E)$, and write

$$\lambda_2(\mathcal{L}_G) = \min_{\substack{x \perp \mathbf{1} \\ x \neq 0}} \frac{\sum_{\{u,v\} \in E} (x_u - x_v)^2}{\sum_{v \in V} x_v^2} \,. \tag{1.2}$$

In addition, for a cycle C, let $\mathcal{B}_R(C)$ be the subgraph induced by all vertices at graph distance at most R from C. We say that C has a *clean radius-R neighborhood* if $\mathcal{B}_R(C)$ is a cycle with disjoint outward trees attached, with no other cycle or chord. Since G is 3-regular, there is one vertex in each outward branch at level 1, two at level 2, and 2^{k-1} at level $k \geqslant 1$. In particular, for a cycle of length ℓ, level k has exactly $\ell 2^{k-1}$ vertices.

Proposition 1.3. *For every $\gamma > 0$ and every $\ell \geqslant 4$ divisible by four, there is a finite integer $R = R(\ell, \gamma)$ with the following property. If a simple cubic graph G satisfies $\lambda_2(\mathcal{L}_G) \geqslant \gamma$ and contains an ℓ-cycle with a clean radius-R neighborhood, then $\mathcal{E}(\theta)$ has a nonsynchronous strict local minimum modulo rotations. Every edge cosine at this local minimum is bounded below by a positive absolute constant.*

Proof of Theorem 1.2 using Proposition 1.3. By [Fri08, Bor15], $\lambda_2(\mathcal{L}_G) \leqslant 2\sqrt{2} + \epsilon \to 1$ for any $\epsilon > 0$. By [Fri08], the graph G has no $(\epsilon \log n)$-tangles for sufficiently small constant ϵ. By [MWW04], the number of ℓ-cycles in G converge in distribution to $\mathrm{Pois}(\frac{2^\ell}{2\ell})$. Thus, G contains an ℓ-cycle with a clean radius-R neighborhood with probability $1 - o(1)$, and therefore satisfies the assumption in Proposition 1.3. Theorem 1.2 then immediately follows. □

1.1. **Statement on AI use.** The results in this note were generated by GPT-6 Astra, prompted by Zhangsong Li without any mathematical input.

2. Proof of Proposition 1.3

The rest part of the note is devoted to the proof of Proposition 1.3. We first need a specific construction for frequencies in a binary tree. For $0 \leqslant t \leqslant \pi/2$, define

$$F(t) = 3t + 2 \sum_{j=1}^{\infty} \arcsin(2^{-j} \sin t).$$

The inverse sine always takes values in $[-\pi/2, \pi/2]$.

Lemma 2.1. *There is a unique $\delta \in (\pi/3, \pi/2)$ such that $F(\delta) = 2\pi$. Put $f = \sin \delta$ and, for every integer $k \geqslant 0$, set*

$$a_k = \sum_{j=k}^{\infty} \arcsin(2^{-j} f).$$

Then

$$2a_0 + \delta = 2\pi, \qquad a_k - a_{k+1} = \arcsin(2^{-k} f), \qquad 0 < a_k \leqslant \pi 2^{-k}\,. \tag{2.1}$$

In particular, $a_0 = \pi - \delta/2$ and $a_R < \delta$ whenever $R \geqslant 2$.

Proof. For $0 \leqslant x \leqslant 1$, convexity of $\arcsin x$ gives $\arcsin x \leqslant \pi x/2$. Hence the series defining F converges uniformly on $[0, \pi/2]$. The function is continuous and strictly increasing, because $3t$ is strictly increasing and every term in the series is nondecreasing. Moreover, $F(0) = 0$, whereas

$$F(\pi/2) = \frac{3\pi}{2} + 2 \sum_{j=1}^{\infty} \arcsin(2^{-j}) > \frac{3\pi}{2} + 2 > 2\pi.$$

Here we used $\arcsin x > x$ for $x > 0$ and $\pi < 4$. The intermediate value theorem and strict monotonicity give a unique root $\delta \in (0, \pi/2)$. Also,

$$F(\pi/3) \leqslant \pi + \pi \sin(\pi/3) = \pi\left(1 + \frac{\sqrt{3}}{2}\right) < 2\pi,$$

so $\delta > \pi/3$. The first identity in (2.1) follows from $F(\delta) = 2\pi$, and the second is telescoping. Finally,

$$a_k \leqslant \frac{\pi}{2} f \sum_{j=k}^{\infty} 2^{-j} = \pi f 2^{-k} \leqslant \pi 2^{-k}.$$

For $R \geqslant 2$, this yields $a_R \leqslant \pi/4 < \delta$. □

To understand the construction, consider a cycle vertex with phase a_0. Give its two cycle neighbors phases a_0 and $-a_0$, and give its outward neighbor phase a_1. The three contributions to its gradient are

$$0, \qquad \sin(2a_0) = -f, \qquad \sin(a_0 - a_1) = f.$$

They cancel. On a binary branch, a vertex at level $k \geqslant 1$ with phase a_k has a parent at a_{k-1} and two children at a_{k+1}. Its gradient is

$$-2^{-(k-1)} f + 2 \cdot 2^{-k} f = 0.$$

The profile with all signs reversed satisfies the same equations. The principal phase difference on an opposite-sign cycle edge is δ, not $2a_0$: indeed $2a_0 = 2\pi - \delta$.

We now finish the proof of Proposition 1.3.

Proof of Proposition 1.3. Let $\delta, f, (a_k)$ be as in Lemma 2.1. Define the absolute constants

$$\eta = \frac{\pi}{2} - \delta, \qquad \rho = \frac{\eta}{4}, \qquad w = \cos(\delta + 2\rho) = \sin(\eta/2) > 0.$$

Choose an integer $R \geqslant 2$ so large that

$$9\sqrt{\ell}\, 2^{-R/2} \leqslant \frac{w\gamma\rho}{4}. \tag{2.2}$$

This choice depends on ℓ and γ, but not on n. The proof follows the following steps.

The approximate phase vector. List the cycle vertices as $v_0, \dots, v_{\ell-1}$ in cyclic order. Set $\sigma_i = 1$ for $i \equiv 0, 1 \pmod 4$ and $\sigma_i = -1$ for $i \equiv 2, 3 \pmod 4$. Give v_i phase $\sigma_i a_0$, and give every vertex at level $k \in \{1, \dots, R\}$ in its outward tree phase $\sigma_i a_k$. Give all remaining vertices phase zero. Denote this real-valued vector by θ^0; phases are interpreted modulo 2π. For $s \in \mathbb{R}$, write $d_{\mathsf{T}}(s, 0) = \min_{q \in \mathbb{Z}} |s - 2\pi q|$. Every edge of G satisfies

$$d_{\mathsf{T}}(\theta^0_u - \theta^0_v, 0) \leqslant \delta < \pi/2. \tag{2.3}$$

Indeed, same-sign cycle edges have difference zero, and opposite-sign cycle edges have principal difference δ by (2.1). An edge from level k to $k+1$, for $0 \leqslant k < R$, has difference $\arcsin(2^{-k} f) \leqslant \delta$. An edge from level R to the exterior has difference $a_R < \delta$. All other edges have difference zero. The clean-neighborhood assumption excludes extra edges inside the displayed trees or between their boundary vertices.

An explicit Euclidean residual estimate. The gradient and Hessian quadratic form are

$$(\nabla \mathcal{E}(\theta))_v = \sum_{u \sim v} \sin(\theta_v - \theta_u),$$

$$x^\top \nabla^2 \mathcal{E}(\theta) x = \sum_{\{u,v\} \in E} \cos(\theta_u - \theta_v)(x_u - x_v)^2.$$

Write $g = \nabla\mathcal{E}(\theta^0)$. The calculations following Lemma 2.1 show that g vanishes on the cycle and on levels $1, \dots, R-1$. At a level-R vertex with sign σ, the two outward neighbors have phase zero, so

$$g_v = \sigma\big(-2^{-(R-1)} f + 2\sin a_R\big), \qquad |g_v| \leqslant (2+2\pi)2^{-R}.$$

There are $\ell 2^{R-1}$ such vertices. If v lies outside $B_R(C)$, let t_v count its neighbors at level R. Then $0 \leqslant t_v \leqslant 3$, $|g_v| \leqslant t_v a_R$, and

$$\sum_{v\notin B_R(C)} t_v = \ell 2^R, \qquad \sum_{v\notin B_R(C)} t_v^2 \leqslant 3\ell 2^R.$$

Consequently,

$$\|g\|_2^2 \leqslant \ell 2^{R-1}(2+2\pi)^2 2^{-2R} + 3\ell 2^R a_R^2 \leqslant \big(2(1+\pi)^2 + 3\pi^2\big)\ell 2^{-R} < 81\ell 2^{-R}.$$

Thus

$$\|g\|_2 \leqslant 9\sqrt{\ell}\, 2^{-R/2}. \tag{2.4}$$

No condition has been imposed on how the exterior neighbors connect to one another, and a vertex outside the ball may be adjacent to several boundary vertices. The factor $t_v^2 \leqslant 3t_v$ explicitly covers this possibility.

Uniform convexity in a fixed ball modulo rotation. Let $\mathcal{H} = \{h \in \mathbb{R}^V : \langle h, \mathbf{1}\rangle = 0\}$ and consider $h \in \mathcal{H}$ with $\|h\|_2 \leqslant \rho$. For each edge, $|h_u - h_v| \leqslant 2\rho$. By (2.3),

$$\cos\big((\theta_u^0 + h_u) - (\theta_v^0 + h_v)\big) \geqslant \cos(\delta + 2\rho) = w.$$

It follows that, for every $x \in \mathcal{H}$,

$$x^\top \nabla^2\mathcal{E}(\theta^0 + h)x \geqslant w x^\top \mathcal{L}_G x \geqslant w\gamma\|x\|_2^2. \tag{2.5}$$

In particular, the lower bound is independent of the graph size.

Existence of an exact stationary point. Minimize $h \mapsto \mathcal{E}(\theta^0 + h)$ on the compact ball $\{h \in \mathcal{H} : \|h\|_2 \leqslant \rho\}$. Taylor's formula and (2.5) imply

$$\mathcal{E}(\theta^0 + h) - \mathcal{E}(\theta^0) \geqslant \langle g, h\rangle + \frac{w\gamma}{2}\|h\|_2^2.$$

On the boundary $\|h\|_2 = \rho$, equations (2.2) and (2.4) therefore give

$$\mathcal{E}(\theta^0 + h) - \mathcal{E}(\theta^0) \geqslant -\frac{w\gamma\rho^2}{4} + \frac{w\gamma\rho^2}{2} = \frac{w\gamma\rho^2}{4} > 0.$$

A minimizer h^* is thus interior. Its gradient in directions orthogonal to $\mathbf{1}$ vanishes. Since $\sum_v (\nabla\mathcal{E}(\theta))_v = 0$ for every θ, the full gradient also vanishes at $\theta^* = \theta^0 + h^*$. The Hessian estimate (2.5) proves that θ^* is a strict local minimum modulo rotations. In fact the same estimate and stationarity show

$$\|h^*\|_2 \leqslant \frac{\|g\|_2}{w\gamma} \leqslant \rho/4.$$

The stationary point is not synchronized. Choose an opposite-sign cycle edge. Its principal difference at θ^0 has magnitude δ, and perturbation by h^* changes that magnitude by at most 2ρ. Hence at θ^* it is at least

$$\delta - 2\rho = \frac{3\delta}{2} - \frac{\pi}{4} > \frac{\pi}{4} > 0,$$

where $\delta > \pi/3$ was used. Therefore θ^* is not constant modulo 2π. As $\lambda_2(L_G) > 0$, the graph is connected and its global minimum energy is zero, attained precisely by synchronized configurations. Our local minimum is not global.

Finally, this construction also gives a representative in the interior of the original domain $[0, 2\pi)^V$. Since $|\theta_v^0| \leqslant a_0 = \pi - \delta/2$ and $\rho < \delta/2$, adding π to every coordinate of θ^* places all coordinates strictly between 0 and 2π. Thus the torus formulation introduces no boundary issue. $\square$

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