Robust Random Graph Matching in Gaussian Models via Vector Approximate Message Passing

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July 3, 2025

38th Annual Conference on Learning Theory

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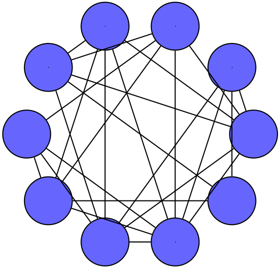

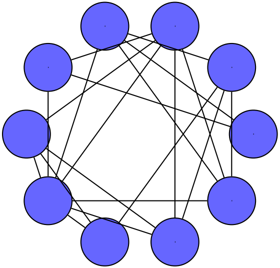

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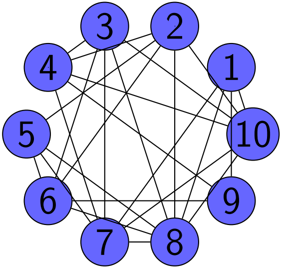

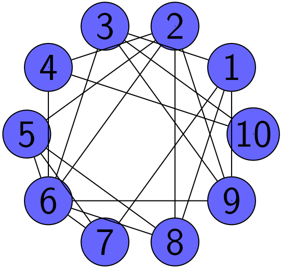

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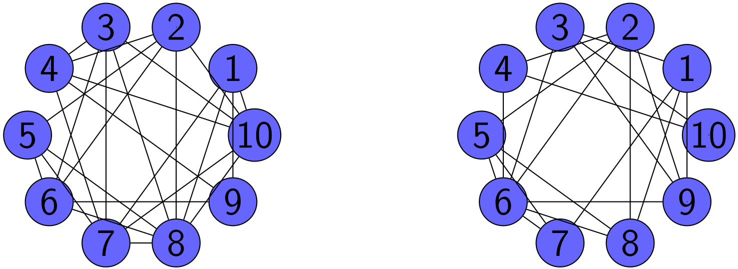

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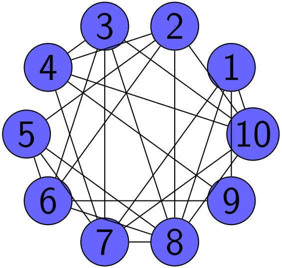

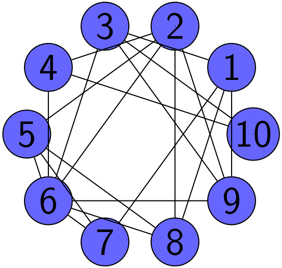

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- NP-hard to solve/approximate in worst case.

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  - Noiseless case ($\rho = 1$): optimal condition is attained in linear-time [Bollobás'82, Czajka-Pandurangan'08].
  - Noisy case ($\rho < 1$): little is known for efficient algorithms until recently.

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- **Motivation from theory**: can we find efficient graph matching algorithms for semi-random models?

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- Observation: the revised matrices $(A', B') = (A + E, B + F)$, where E, F supported on an unknown $\epsilon n * \epsilon n$ principle minor of (A, B).

Our result: a robust Gaussian matching algorithm

ρ: edge correlation; ϵn: size of corruption; π_*: hidden matching.

Theorem (L.'25)

Exact recovery is achieved efficiently by an approximate message passing algorithm w.h.p. if

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- The first graph matching algorithm that is robust under $n^{1-o(1)}$ size of perturbations.
- Extends to the case of correlated Erdős-Rényi models when the edge-density q is a constant.

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[Ivkov-Schramm'25]: using a spectral cleaning procedure, AMP algorithms can be performed robustly.

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- Hope: if we choose a suitable denoiser function φ and $\Xi^{(t)}$, then at a large time t^* we will have

$$\Pi_* = \arg\max_{\Pi \in \mathfrak{S}_n} \langle f^{(t^*)}, \Pi g^{(t^*)} \rangle,$$

then we can find Π_* by solving a linear assignment problem.

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<u>References</u>:

Zhangsong Li. Robust random graph matching in Gaussian models via vector approximate message passing. arXiv:2412.16457v2.