Recent Progress on Random Graph Matching Problems

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Based on a joint work with J.Ding

Application 1: Network de-anonymization

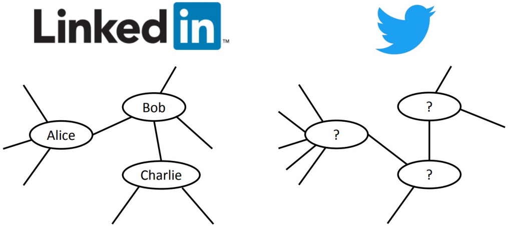

Figure 1: Picture courtesy of R.Srikant

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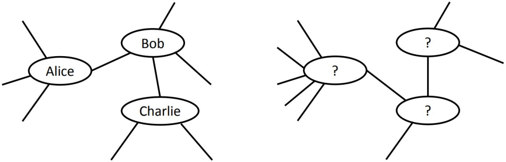

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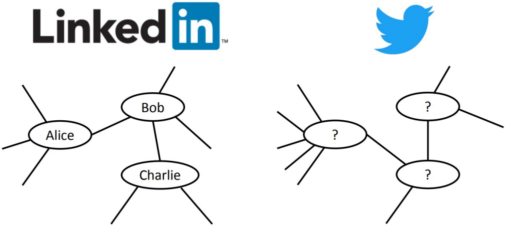

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- Successfully de-anonymize Netflix by matching it to IMDB. [Narayanan-Shmatikov '08]
- Correctly identified 30.8% of node mappings between Twitter and Flickr. [Narayanan-Shmatikov '09]

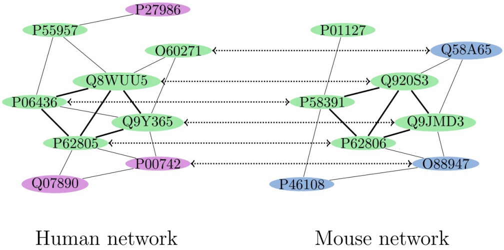
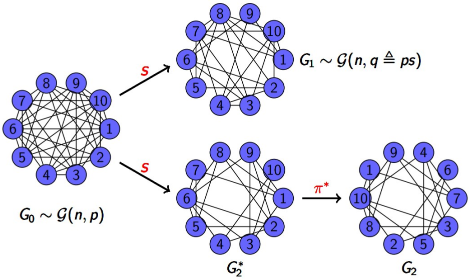


Figure 2: [Kazemi-Hassani-Grossglauser-Modarres '16]

- Ontology: Discover proteins with similar functions across different species based interaction network topology.

An idealized model: correlated Erdős-Rényi graphs



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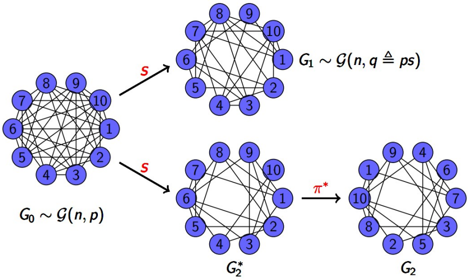

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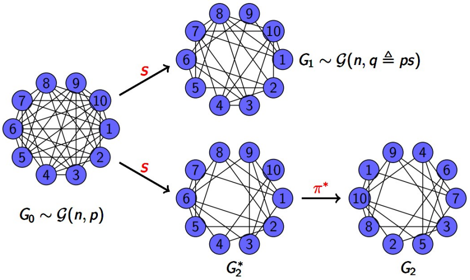

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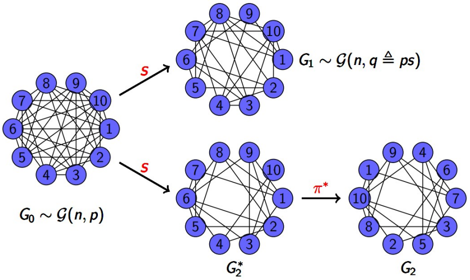

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- Advantage: simple probabilistic model; suitable playground for developing mathematical theory.
- Disadvantage: almost all realistic networks are not Erdős-Rényi.

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- Reduced the problem into the network alignment problem of correlated random graphs.
- Unfortunately, the classical graph alignment problem is a NP-hard optimization problem, so we must seek help from randomness.

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- Results: determine the exact information threshold for exact recovery; Determine the information threshold for partial-recovery and detection in the dense region ($p = n^{o(1)}$) exactly and in the non-dense region ($p = n^{-\alpha+o(1)}$ where $0 < \alpha < 1$) up to constants.

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- [Ding-Du'22+a,22+b]: determine the exact information threshold for detection and partial-recovery in the non-dense region via a modified statistics based on densest subgraphs.

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- "Optimization relaxation" based algorithm:
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- All the above algorithms either run in pseudo-polynomial time (i.e. $n^{O(\log n)}$) or succeeds only when the correlation approaches 1 (with rate polylog n).

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- [Mao-Wu-Xu-Yu'22+]: poly-time algorithm when correlation $> \sqrt{\text{Otter's constant}}$, based on a carefully curated family of rooted trees called chandeliers (substantially improving MRT21+, and covers much wider parameter regime).

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  - Might shed lights on many matching problems too.
- An ongoing work with J. Ding: A polynomial time iterative algorithm for random graph matching with non-vanishing correlation.

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  - Major challenge 1: propose models with general applicability where theorists can say something meaningful.
  - Major challenge 2: propose models for important scientific problems worth extensive theoretic study.

References

1. [BCL+19] B. Barak, C.-N. Chou, Z. Lei, T. Schramm, and Y. Sheng. (nearly) efficient algorithms for the graph matching problem on correlated random graphs. In Advances in Neural Information Processing Systems, volume 32. Curran Associates, Inc., 2019.
2. [DCKG19] O. E. Dai, D. Cullina, N. Kiyavash, and M. Grossglauser. Analysis of a canonical labeling algorithm for the alignment of correlated erdos-rényi graphs. Proc. ACM Meas. Anal. Comput. Syst., 3(2), jun 2019.
3. [DD22+a] J. Ding and H. Du. Detection threshold for correlated erdos-renyi graphs via densest subgraph. IEEE Trans. Inform. Theory, online 2023.
4. [DD22+b] J. Ding and H. Du. Matching recovery threshold for correlated random graphs. Preprint, arXiv:2205.14650.
5. [DL22+] J. Ding and Z. Li. A polynomial time iterative algorithm for matching Gaussian matrices with non-vanishing correlation. Preprint, arXiv:2212.13677.
6. [DMWX21] J. Ding, Z. Ma, Y. Wu, and J. Xu. Efficient random graph matching via degree profiles. Probab. Theory Related Fields, 179(1-2):29–115, 2021.
7. [GM20] L. Ganassali and L. Massouli´e. From tree matching to sparse graph alignment. In J. Abernethy and S. Agarwal, editors, Proceedings of Thirty Third Conference on Learning Theory, volume 125 of Proceedings of Machine Learning Research, pages 1633–1665. PMLR, 09–12 Jul 2020.
8. [GMS22] L. Ganassali, L. Massouli´e, and G. Semerjian. Statistical limits of correlation detection in trees. arXiv:2209.13723.

References+

9. [MRT21+] C. Mao, M. Rudelson, and K. Tikhomirov. Exact matching of random graphs with constant correlation. Preprint, arXiv:2110.05000.
10. [MWXY22+] C. Mao, Y. Wu, J. Xu, and S. H. Yu. Random graph matching at Otter's threshold via counting chandeliers. Preprint, arXiv:2209.12313.
11. [MWXY21+] C. Mao, Y. Wu, J. Xu, and S. H. Yu. Testing network correlation efficiently via counting trees. Preprint, arXiv:2110.11816.
12. [NS08] A. Narayanan and V. Shmatikov. Robust de-anonymization of large sparse datasets. In 2008 IEEE Symposium on Security and Privacy (sp 2008), pages 111–125, 2008.
13. [NS09] A. Narayanan and V. Shmatikov. De-anonymizing social networks. In 2009 30th IEEE Symposium on Security and Privacy, pages 173–187, 2009
14. [WXY21+] Y. Wu, J. Xu, and S. H. Yu. Settling the sharp reconstruction thresholds of random graph matching. Preprint, arXiv:2102.00082.

Thank you!