Asymptotic Diameter of Preferential Attachment Model

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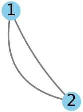
Joint work with Hang Du (MIT) and Haodong Zhu (TU/E)

YMSC Probability Seminar

Preferential attachment model (PAM)

At each time t, a new vertex labeled t arrives and forms m edges, one at a time, to existing nodes $v \in [t-1]$:

$$\mathbb{P}(t \to v) \propto \deg(v) + \delta \text{ where } \delta > -m\,.$$



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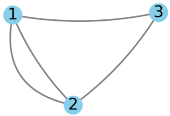

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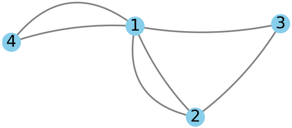

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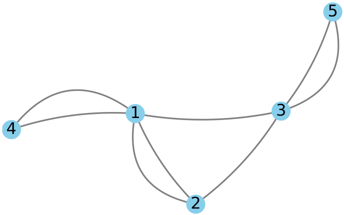

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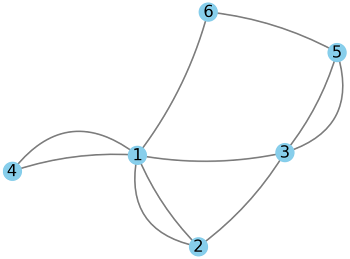

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- The smaller δ, the stronger preference for high-degree vertices
- A popular dynamical model that shares many similar features as in empirically studied real-world networks.

Features of PAM: Power-law degree distribution

Theorem (Bollobás-Riordan-Spencer-Tusnády'01, Deijfen-van den Esker-van der Hofstad-Hooghiemstra'09)

PAM with parameter m, δ *yields power-law degree sequence with exponent* $\tau = 3 + \delta/m > 2$.

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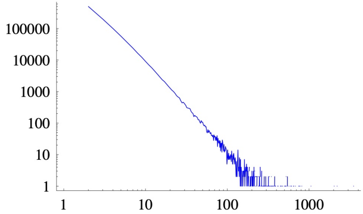

Figure: degree sequences in PAM with $m = 2, \delta = 0, \tau = 3, n = 10^6$ (picture courtesy of Remco van der Hofstad)

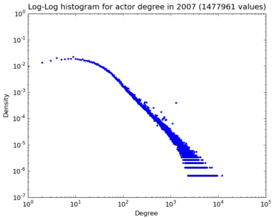

Figure: degree sequences in Internet Movie Data Base 2007 [Britton-Deijfen-Lőf'2007]

Features of PAM: Small world phenomenon

Figure: Six degrees of separation: "Everybody on this planet is separated only by six other people".

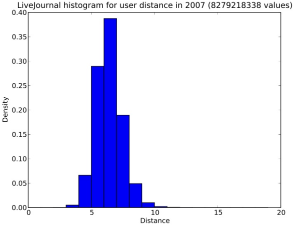

Figure: Distances in social networks *Livejournal* [Backstrom-Boldi-Rosa-Ugander-Vigna'2010]

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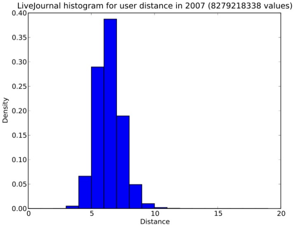

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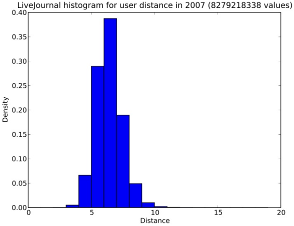

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- Question: Can we rigorously justify the small world phenomenon in PAM?
- Equivalently, does PAM have small diameters?

Previous results on diameters of PAM

average degree: $2m$; $\mathbb{P}(t \to v) \propto \deg(v) + \delta$;

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where $\theta \in (0, 1)$ is the solution to $\theta + (1 + \delta)(1 + \log \theta) = 0$.

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- Remaining case: PAM with $m \geq 2, \delta > 0$.

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- [Dommers-van der Hofstad-Hooghiemstra'10]: the diameter of PAM with $m \geq 2, \delta > 0$ is typically $O(\log n)$.

Typical distance of PAM with $m \geq 2, \delta > 0$

average degree: $2m$; $\mathbb{P}(t \to v) \propto \deg(v) + \delta$; $\mathrm{PA}_n^{(m,\delta)}$: law of PAM

Theorem (van der Hofstad-Zhu'25+)

Let ν to be the exponential growth parameter of the local limit of the preferential attachment model, then

$$\mathbb{P}_{G \sim \mathrm{PA}_n^{(m,\delta)}} \mathbb{P}_{u,v \sim \mathrm{unif}(V(G))} \left(\mathrm{dist}_G(u,v) = (1 + o(1)) \log_\nu n \right) = 1 - o(1)\,,$$

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- Conjecture in [van der Hofstad-Zhu'25+]: typically the diameter of PAM with $m \geq 2, \delta > 0$ is also $(1+o(1))\log_\nu n$.

Our result: from typical distance to diameter

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- Conclusion: typically $\mathrm{diam}(G) \leq (1 + o(1)) \log_\nu n$.

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It seems that our result

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  - [Riordan-Wormald'08]: $M_n = c(\lambda)\log n$;

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- Giant component of Erdős-Rényi graph with average degree $\lambda = 1 + \Omega(1)$:
  - [Riordan-Wormald'08]: $M_n = c(\lambda) \log n$;
  - $R_n = \Theta(1) \cdot \log n$;
  - [Fernholz-Ramachandran'07] (see also [Ding-Kim-Lubetzky-Peres'10] for more general λ): diameter $= (1 + \Theta(1)) \cdot$ average distance.

Proof idea

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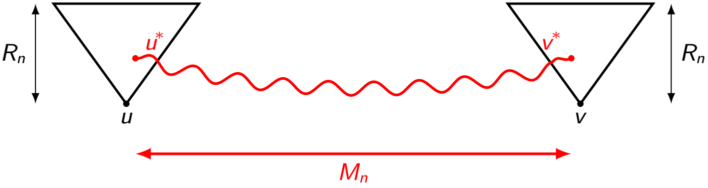

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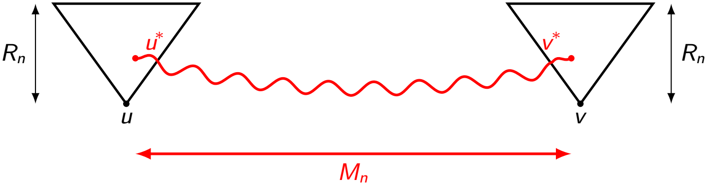

- Diameter at most $M_n + 2R_n$. $M_n = \log_\nu n$, $R_n = o(\log n)$.

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Taking $R_n = (\log n)^{2/3}$,

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  - $m \geq 2$ condition $\Rightarrow$ exponential growth in neighborhood size.
- Major Challenge: dealing with dependence issue.

Lemma (Conditional attachment lemma)

Let E be a set of potential edges in $G_n \sim \mathrm{PA}$ and A be a set of vertices. Assume that $A \subset [s, n]$, then

$$\mathrm{PA}[u \to A \mid E \subset E(G_n)] \leq \frac{|A|(m + \delta + 1) + |E|}{(2s - 2)m + s\delta}\,.$$

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- $\Rightarrow \mathrm{PA}(\mathcal{G}_1) = 1 - o(1)$ under PA.

Bridging the gap by sprinkling

- Breaking $[1, n]$ into three sets: $G_{n-2K_n} \triangleq [1, n-2K_n]$, $I_1 \triangleq [n-2K_n, n-K_n]$ and $I_2 \triangleq [n-K_n, n]$ where $K_n = n/\log n$.
- There exists a w_1 in I_1, such that $w_1 \to$ a typical vertex and $w_1 \to N_{R_n}(u)$, with probability $1 - \left(1 - O((\log n)^4/n)\right)^{K_n} = 1 - \exp\left(-O((\log n)^3)\right)$.
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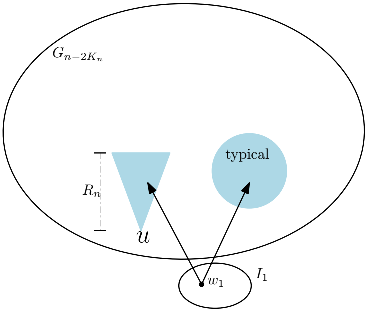

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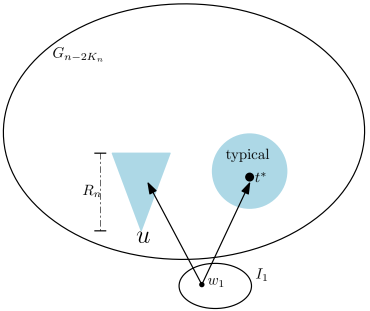

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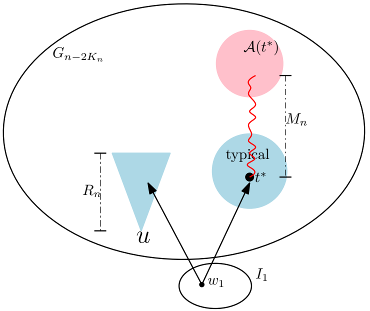

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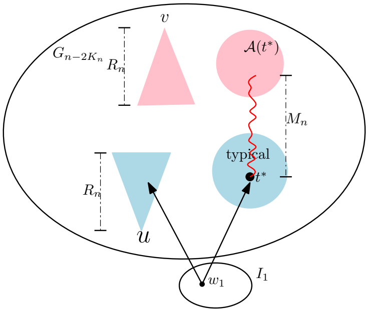

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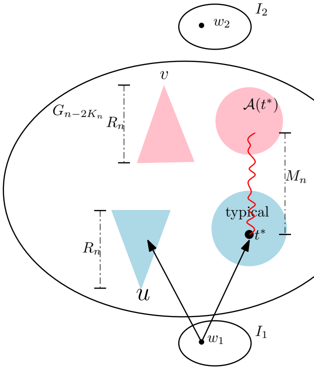

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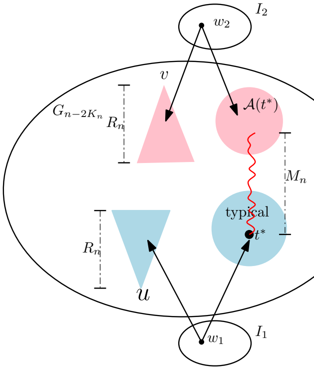

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- Show for any $u \in [n - 2K_n, n]$, $\mathrm{dist}_{G_n}(u, [1, n - 2K_n]) \le R_n + 2$.
- BFS (breath-first search) in $N_{R_n}(u)$. $\mathcal{F}_{k-1}$ as the attachment of $v_1, \dots, v_{k-1}$.
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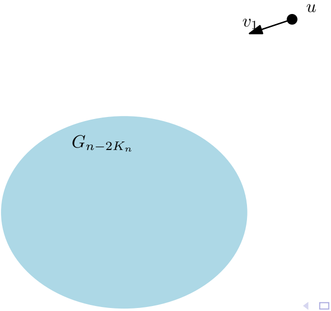

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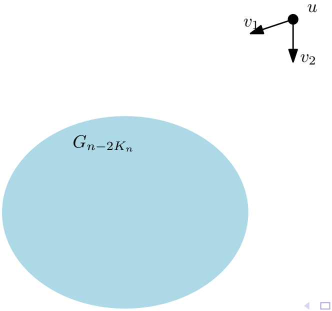

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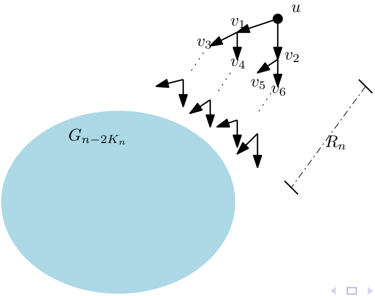

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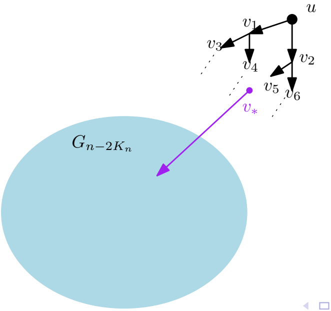

Outlook and discussions

- We prove the asymptotic diameter of the PA model is $\log_\nu n$ when $m \geq 2, \delta > 0$.
- End of the story? We hope the proof technique can be applied to other graph models.
- Open question:
  (1) Conditional on diameter being $C \log_\nu n$ with $C > 1$, what is the graph structure?
  (2) Pinpointing the second order of the diameter of PA model. Conjecture: $\log_\nu n + O(\log \log n)$.

Thank you!