Low-degree Hardness for Detection in Correlated Erdős-Rényi Graphs

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Based on a joint work with J. Ding (PKU) and H. Du (MIT)

Application: network de-anonymization

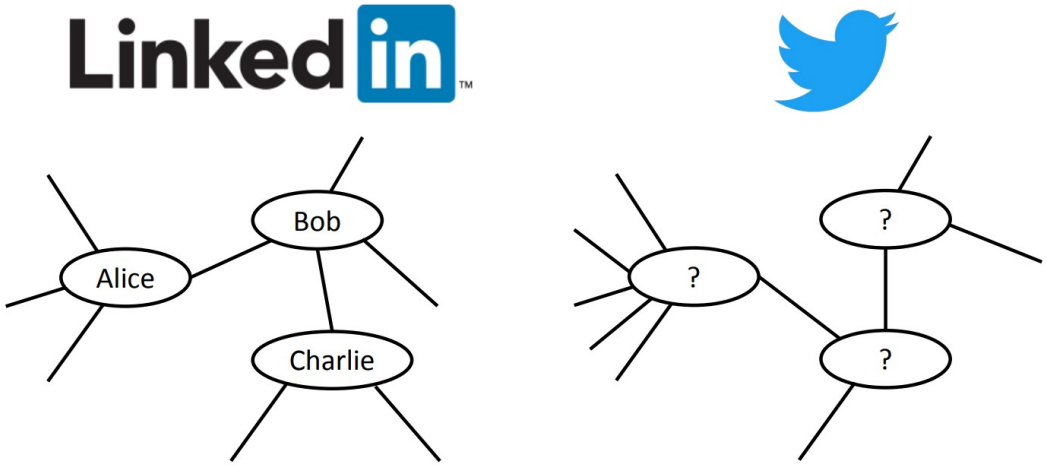

Figure 1: Picture courtesy of R.Srikant

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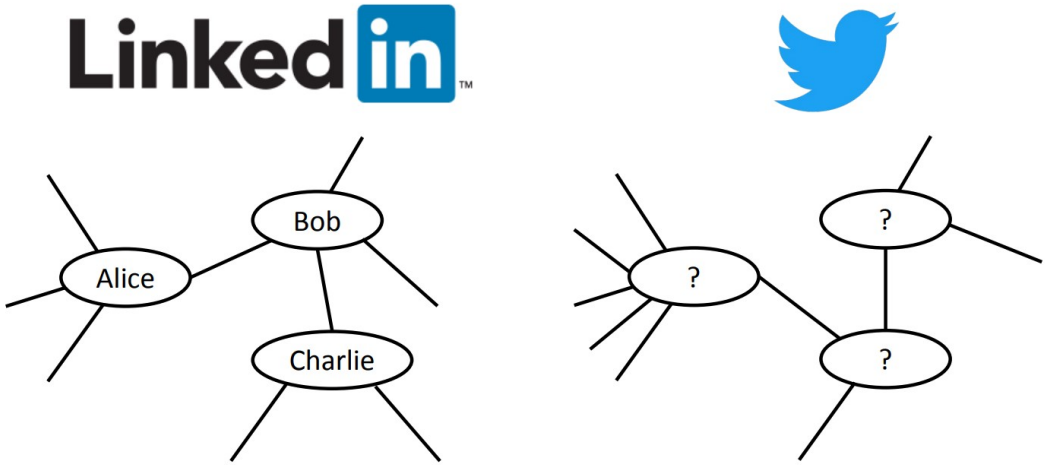

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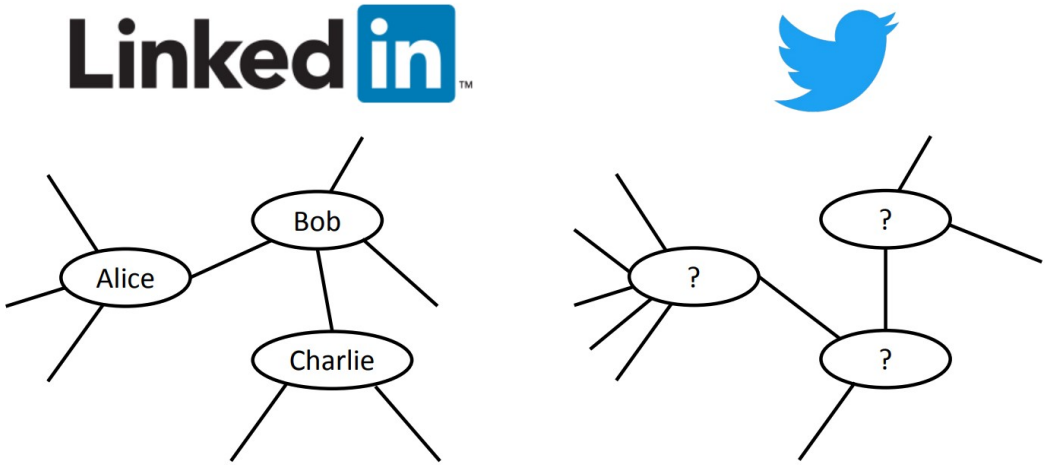

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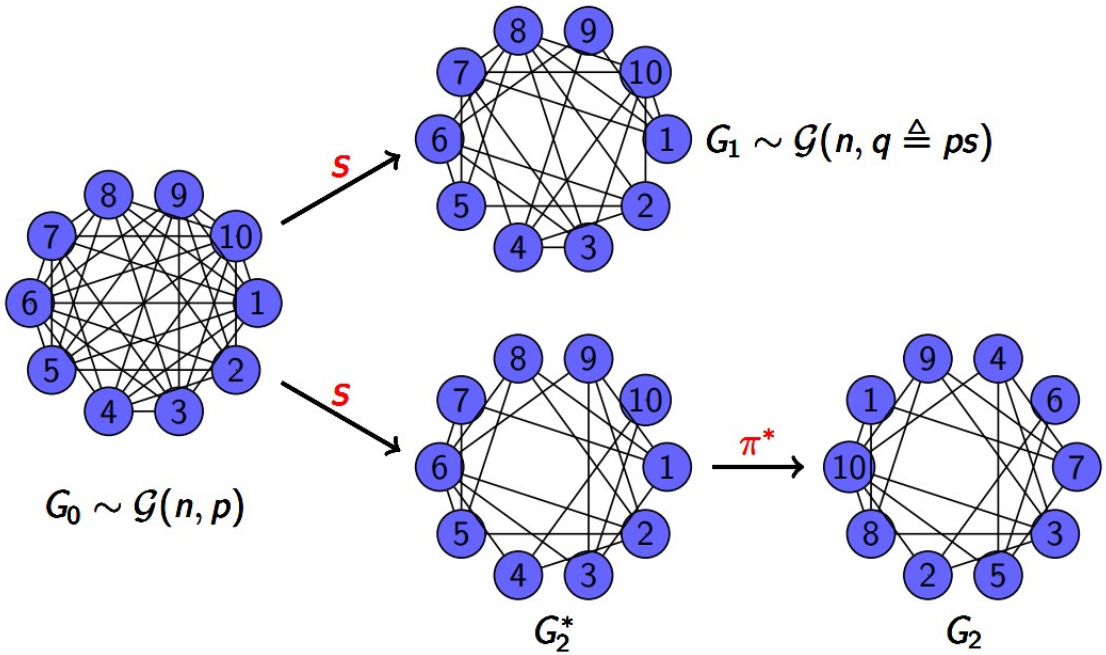
- Successfully de-anonymize Netflix by matching it to IMDB. [Narayanan-Shmatikov '08]
- Correctly identified 30.8% of node mappings between Twitter and Flickr. [Narayanan-Shmatikov '09]

An idealized model: correlated Erdős-Rényi graphs



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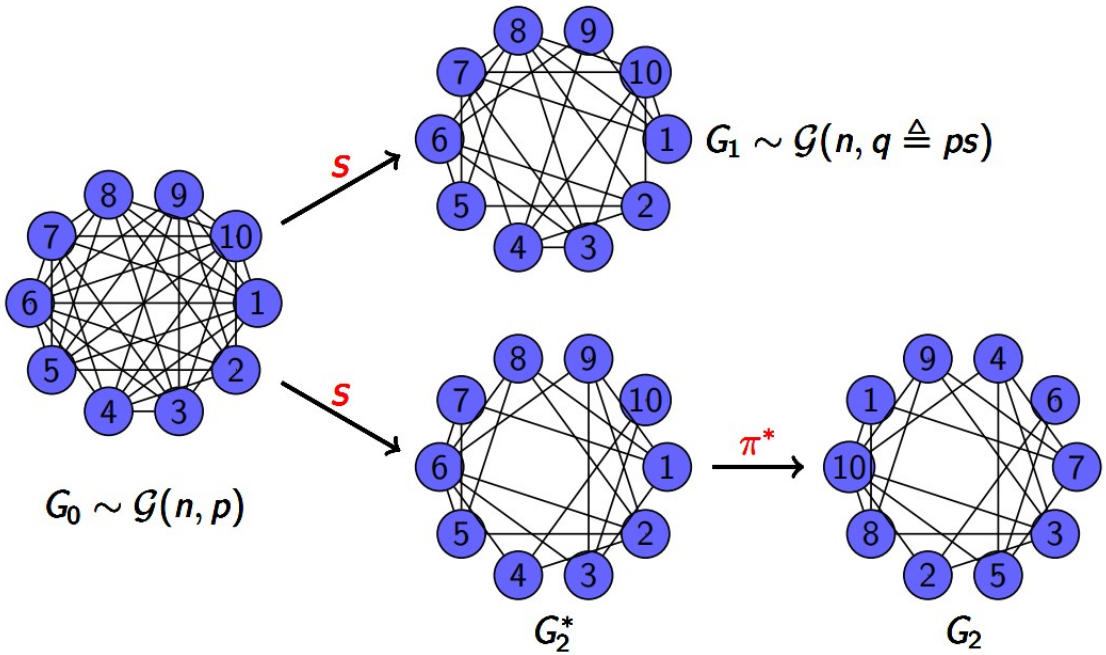

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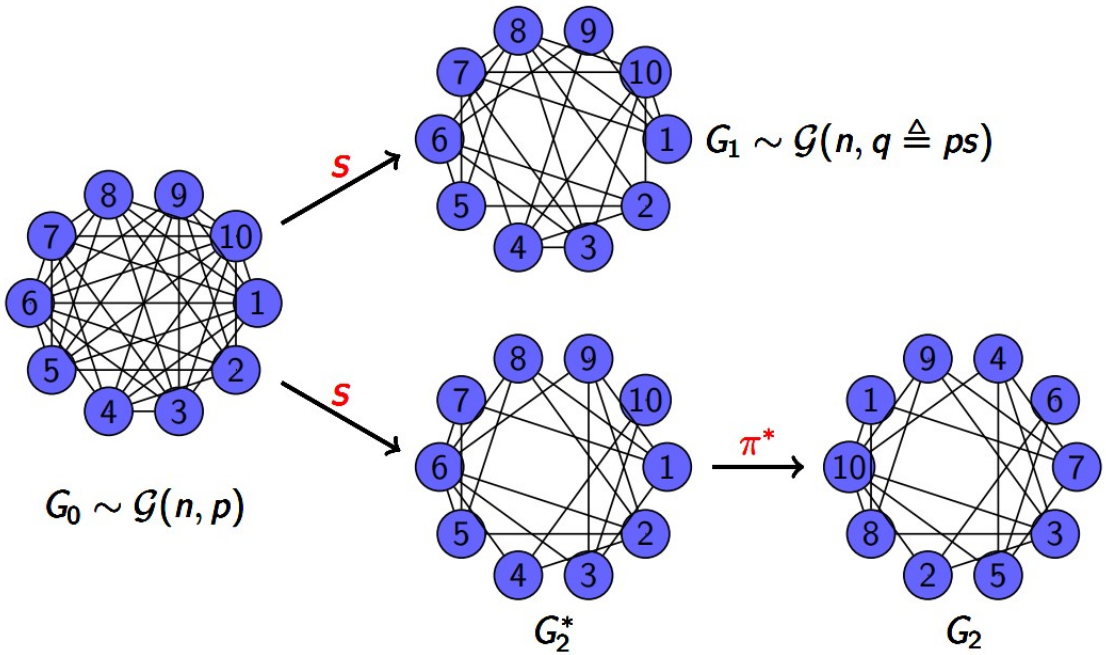

- G_1 and G_2 are Erdős-Rényi graphs with edge-density $q = ps$, and edge-correlation $\rho = \frac{s(1-p)}{1-ps}$.
- Two basic problems regarding this model: (1) the detection problem, i.e., testing $\mathbb{P}$ against the law of two independent Erdős-Rényi graphs on with edge density q; (2) the matching problem, i.e., recovering the latent matching π_* from the adjacency matrices (A, B) of (G_1, G_2).

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- [Ding-Du '23a,23b]: determine the exact information threshold for detection and partial-recovery in the non-dense region via a modified statistics based on densest subgraphs.

Efficient algorithms

- Progressively improved algorithms have been obtained by commmunity (e.g. [Dai-Cullina-Kiyavash-Grossglauser '18], [Barak-Chou-Lei-Schramm-Sheng '19], [Ding-Ma-Wu-Xu '21], [Mao-Rudelson-Tikhomirov '21], etc.)

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  - [Mao-Wu-Xu-Yu '21+,23]: polynomial time algorithm for detection/matching when $q > \frac{\log n}{n}$ and correlation $\rho > \sqrt{\alpha}$ where $\alpha \approx 0.338$ is the Otter's constant, based on counting carefully curated family of rooted trees.

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  - [Ding-L. '22+,23+]: polynomial time iterative algorithm for matching when $p \geq n^{-1+\delta}$ and correlation ρ non-vanishing.

Information-computation gap

regime	Info, Detection	Info, ExaMatch	Alg
$q = n^{o(1)}$	$\rho^2 > \frac{\log n}{nq \log q^{-1}}$	$\rho^2 > \frac{\log n}{nq \log q^{-1}}$	$\rho = \Omega(1)$
$q = n^{-1+c+o(1)}$	$\rho^2 > \frac{\lambda_*}{nq}$	$\rho^2 > \frac{\log n}{nq}$	$\rho = \Omega(1)$
$q = n^{-1+o(1)}$	$\rho^2 > \left(\frac{1}{nq} \wedge u_0\right)$	$\rho^2 > \frac{\log n}{nq}$	$\rho^2 > \alpha$

- λ_* (resp. u_0) are constants that can be determined in [DD23a] (resp. [WXY23]); α is the Otter's constant.

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- Information-computation gaps: a major challenge in many random combinatorial optimization problems.
- Question: how to give lower bounds on computational complexity for problems with random input?

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  - Approximate message passing (AMP)

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  - Low-degree polynomials fail in the conjectured "hard" regime.
- [Hopkins'18] "Low-degree conjecture" (informal): for "natural" problems, the failure of degree-D polynomials implies the failure of all algorithms with running time $e^{\widetilde{\Theta}(D)}$.

Our Results

Theorem (Ding-Du-L., informal)

There is evidence suggesting that algorithms based on polynomials of degree $O(\rho^{-1})$ fail for detection in the correlated Erdős-Rényi graph model.

Furthermore, if q, ρ satisfies $nq = n^{o(1)}$ and $\rho^2 < \alpha - \varepsilon$ for some arbitrary constant $\varepsilon > 0$ (where $\alpha \approx 0.338$ denotes the Otter's constant), then there is evidence suggesting that algorithms based on polynomials of degree D fail for detection as long as

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- Also suggest that the (exact) graph matching problem is computationally hard in the aforementioned regimes.
- In particular, suggest that the algorithms in [MWXY21+,23] [DL22+,23+] have nearly reached the limit of efficient algorithms.

Comparison between algorithms and hardness

	Algorithms	Hardness
Detection dense regime	$\rho = \Omega(1); D = O(1)$ [DL23+]	$\rho = o(1); D = O(\rho^{-1})$
Detection sparse regime	$\rho > \sqrt{\alpha}; D = O(1)$ [MWXY21+]	$\rho < \sqrt{\alpha}; \log D = o(d(n,q))$
ExaMatch dense regime	$\rho = \Omega(1); D = O(1)$ [DL23+]	$\rho = o(1); D = O(\rho^{-1})$
ExaMatch sparse regime	$\rho > \sqrt{\alpha}; D = O(\log n)$ [MWXY23]	$\rho < \sqrt{\alpha}; \log D = o(d(n,q))$

- $d(n,q) = \frac{\log n}{\log nq} \wedge \sqrt{\log n}$.
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- Look for a degree-D polynomial $f : \mathbb{R}^{n*n} \otimes \mathbb{R}^{n*n} \to \mathbb{R}$ such that f is "large" under $\mathbb{P}$ and "small" under $\mathbb{Q}$.

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- Key items: signal-to-noise ratio

$$\mathrm{SNR}_{\leq D} = \max_{\deg(f) \leq D} \frac{\mathbb{E}_{\mathbb{P}}[f]}{\sqrt{\mathbb{E}_{\mathbb{Q}}[f^2]}} \qquad \frac{\text{mean in } \mathbb{P}}{\text{fluctuation in } \mathbb{Q}}$$

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  - $\mathrm{SNR}_{\leq D} = O(1)$: degree-$D$ polynomials fail.

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- Since $\mathbb{Q}$ has independent entries, we can directly calculate the projection using orthogonal polynomials w.r.t. $\mathbb{Q}$.

Warm up: the proof in dense regime

- For two graphs S_1, S_2 with no isolated vertices, define the polynomial ϕ_{S_1,S_2} by

$$\phi_{S_1,S_2}\left(\{A_{i,j}\},\{B_{i,j}\}\right) = \prod_{(i,j)\in E(S_1)} \overline{A}_{i,j} \prod_{(i,j)\in E(S_2)} \overline{B}_{i,j},$$

where $\overline{A}_{i,j} = \frac{A_{i,j}-q}{\sqrt{q(1-q)}}, \overline{B}_{i,j} = \frac{B_{i,j}-q}{\sqrt{q(1-q)}}$.

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- $\{\phi_{S_1,S_2} : |E(S_1)| + |E(S_2)| \le D\}$ constitutes a standard orthogonal basis.
- It can be directly calculated that

$$\left\|L_{\le D}\right\|^2 = \sum_{|E(S_1)|+|E(S_2)|\le D} \left(\mathbb{E}_{\mathbb{P}}[\phi_{S_1,S_2}]\right)^2$$

and

$$\mathbb{E}_{\mathbb{P}}[\phi_{S_1,S_2}] = \begin{cases} \rho^{|E(S_1)|} \cdot \frac{\mathrm{Aut}(S_1)}{n(n-1)\cdots(n-|V(S_1)|+1)}, & \text{if } S_1 \cong S_2, \\ 0, & \text{otherwise.} \end{cases}$$

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The problem reduces to controlling

$$\sum_{S_1 \cong S_2, |E(S_1)|+|E(S_2)| \leq D} \rho^{2|E(S_1)|} \cdot \frac{\mathrm{Aut}(S_1)^2}{[n(n-1)\cdots(n-|V(S_1)|+1)]^2}$$

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- It is $O(1)$ provided that $D = O(\rho^{-1})$, using the fact that

$$\#\{|E(\mathbf{H})| = k\} \le \sum_{\ell=1}^{2k} \#\{|E(\mathbf{H})| = k, |V(\mathbf{H})| = \ell\} \le \sum_{\ell=1}^{2k} \binom{\ell(\ell-1)/2}{k}$$

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- However, since $q = n^{-1+o(1)}$, with high probability all subgraphs (with at most D edges) of (G_1, G_2) should have "low" edge-density.
- This inspired us to work with a truncated version of $\mathbb{P}$ rather than $\mathbb{P}$ itself.

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- Define $L' = \mathrm{d}\mathbb{P}'/\mathrm{d}\mathbb{Q}$, we still have

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  4. We can show the number of admissible graphs with k edges is approximately α^{-k} where α is the Otter's constant, thus $(*) = O(1)$ when $\rho < \sqrt{\alpha}$.

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- We can control $\mathbb{E}_{\mathbb{P}|\pi_*(1)=1}[f]$ in the similar manner but we don't know how to control $\mathbb{E}_{\mathbb{P}}[f^2]$.

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Thank you!